

Constant locked inertia tensor trajectories for simple mechanical systems with symmetry

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Abstract

Saari's Conjecture, generalized from its usual context of the N -body problem to a simple mechanical system with symmetry, says roughly that a condition of constant locked inertia tensor (interpreted appropriately) along a solution curve should guarantee that the curve is a relative equilibrium. Using a local Lagrangian slice parametrization about a non-symmetric point in phase space, we offer the motion in the form of a reduced Euler–Poincaré–Lagrange system together with the reconstruction equation. We state necessary and sufficient conditions for the existence of relative equilibria in this parametrization. These conditions allow us to relate curves with constant locked inertia tensors to relative equilibria. We find a class of simple mechanical systems with symmetry for which Saari's Conjecture is true. We also show that if a simple mechanical system with n degrees of freedom is symmetric under the free linear action of a k -dimensional Lie group where $k(k+1)/2 \geq (n-k)$, then a version of Saari's Conjecture holds except at specific isolated points. We apply our results to the three-dimensional three-body and four-body problems and to the n -dimensional general relative two-body problem.

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1. Introduction

A relative equilibrium for a mechanical system with symmetry is a solution of the equations of motion that is also the orbit of a one-parameter symmetry group. For the planar N -body problem of celestial mechanics a relative equilibrium is a solution in which the whole system rotates with a constant angular velocity about a fixed axis through the center of mass. Such a configuration has a moment of inertia that is constant in time. Saari [17] conjectured that a solution of the N -body problem has a constant moment of inertia if and only if it is a relative equilibrium. Renewed interest in Saari's Conjecture is reflected in a number of recent papers on the N -body problem.

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The concept of the inertia tensor in the N -body problem generalizes to that of the *locked inertia tensor* in simple mechanical systems with symmetry. Because the locked inertia tensor is an important ingredient in stability theory for a simple mechanical system with symmetry (see, for example, [22]), Saari's Conjecture in this more general context is a natural question. The first step in this direction was an analysis of a generalization of Saari's Conjecture to mechanics on Lie groups [5]. Here we shall investigate the conjecture for even more general simple mechanical systems with symmetry.

The N -body problem consists of particles with masses m_A , $1 \leq A \leq N$, at positions $q_A(t) \in \mathbb{R}^n$ relative to a fixed inertial frame, interacting by pairwise mutual Newtonian ($-1/R$) potentials. Saari's Conjecture states that a solution of the N -body problem has constant moment of inertia if and only if the system is in relative equilibrium. For the system to be in relative equilibrium, necessity of the condition of constant moment of inertia is obvious. The profound piece of Saari's Conjecture is sufficiency.

Recent interest in Saari's Conjecture is partly the indirect result of the discovery of the "figure eight" periodic solution to the three-body problem, numerically by Moore [12] and analytically by Chenciner and Montgomery [2]. One of the observations made was that the figure eight displays an almost constant moment of inertia. In [21] Simó proved numerically that the moment of inertia along this solution is indeed not constant and therefore Saari's Conjecture remained open.

Subsequently, Saari's Conjecture has been proven for the planar three-body problem. (See [6,10,11].) Diacu et al. [4] proved Saari's Conjecture for the case of N collinear bodies, but the general conjecture is still unsolved for $N \geq 4$. By altering the potential function Santoprete [18] has produced a counterexample to a generalized Saari's Conjecture. There are other counterexamples in the literature, for instance that of Roberts [15] concerning a set-up with negative masses. Schmah and Stoica [20] have taken a different approach, showing that, given an arbitrary function (without too many critical points) and within the class of smooth vector fields with a given free symmetry, generic vector fields have no non-relative-equilibrium solutions conserving the function. In the light of their results, it is to be *expected* that the relative equilibria are the only solutions with constant moment of inertia.

At a meeting in 2002, J. Marsden hypothesized that Saari's Conjecture should admit an extension to more general mechanical systems with symmetry. Chenciner [1] later asked: "Is there a conceptual proof for Saari's Conjecture? Why not fix the moment of inertia tensor and ask the same question (maybe in higher dimensions)?" Thus we are motivated toward such possible extensions of the conjecture.

For a manifold Q and a Lie group G acting freely and properly on the left on Q , a *Lagrangian simple mechanical system with symmetry* consists of a real-valued Lagrangian on the tangent bundle TQ that is in the form of kinetic energy minus potential energy and invariant under the tangent lifted action. For a simple mechanical system with symmetry, it would appear that the moment of inertia is the locked inertia tensor, leading to the following *naive generalization of Saari's Conjecture*.

A Lagrangian simple mechanical system with symmetry is at a point of relative equilibrium if and only if the locked inertia tensor is constant along the integral curve that passes through that point.

In the planar N -body problem, this naive conjecture reduces to Saari's original conjecture with potentials not necessarily Newtonian. However, the naive conjecture is false due to the aforementioned counterexample described in [18]. Moreover, Hernández-Garduño et al. [5] found a counterexample in the setting of mechanics on Lie groups where the locked inertia tensor is not constant along a relative equilibrium. In other words, the direction that was immediately true for the planar N -body problem fails, and one has to take more care when interpreting the moment of inertia. We defer to Section 2 the discussion about an appropriate formulation of Saari's Conjecture in the context of simple mechanical systems.

Rather than try to prove or disprove the conjecture, we focus upon finding geometric relations between the locked inertia tensor and relative equilibria. More precisely, assuming a constant locked inertia tensor, we deduce necessary and sufficient conditions for relative equilibria. We conclude that the solution set with constant locked inertia tensor may include other solutions besides relative equilibria. We also show that under some restrictions related to the dimensions of the group, a generalized Saari's Conjecture holds for linear actions. This result is applied to the spatial three- and four-body problems.

The key to our investigation is expressing the dynamics in slice coordinates. Locally near a non-symmetric point $q_0 \in Q$, we may write $TQ \simeq G \times (\mathfrak{g} \times TS)$, where \mathfrak{g} is the Lie algebra of G , and S is a submanifold of Q orthogonal at q_0 to the group orbit. Then the dynamics splits into motion along the group (the reconstruction equation) coupled

to motion in the reduced space $\mathfrak{g} \times TS$ (the slice equations). Since we deduce the dynamics in the reduced space via a variational principle, we call the reduced system the *Euler–Poincaré–Lagrange* (EPL) equations. In the (locally) reduced space, the equations form a coupled system of a generalized rigid body on \mathfrak{g} and a vibrational non-symmetric Lagrangian on TS . For a simple mechanical system, a relative equilibrium corresponds to a fixed velocity on \mathfrak{g} and a critical point of the augmented potential. Throughout the paper we assume symmetries given by compact matrix Lie groups. The configuration space Q is assumed to be finite dimensional.

The flow of this paper is as follows. In Section 2 we briefly review the theory of simple mechanical systems with symmetry and state the Refined Saari Problem in this context. This generalization agrees with the formulation for the particular case of mechanics on Lie groups described in [5]. The next section describes a slice parametrization around a non-symmetric point and introduces Lagrangian slice coordinates in the neighborhood of a non-symmetric point. Next appears a section on the Lagrangian in slice coordinates, the local reduction in a slice parametrization, and the EPL equations. The next section presents necessary and sufficient conditions for relative equilibria in slice parametrization. In Section 4 we discuss the relation between relative equilibria and constant inertia solutions. We revisit the Refined Saari Problem in a slice parametrization and close with some applications.

2. Simple mechanical systems and Saari’s Conjecture

2.1. Simple mechanical systems with symmetry

This section recalls the definitions of the locked inertia tensor and a relative equilibrium in the context of a generic simple mechanical system with symmetry. This exposition borrows both Marsden’s notation and his results [7]. We begin with some general facts about Lagrangian simple mechanical systems.

A *Lagrangian simple mechanical system* on a configuration manifold Q consists of a Lagrangian $L : TQ \rightarrow \mathbb{R}$ of the form kinetic energy minus potential energy, where the potential energy is a differentiable function on Q denoted by $V(q)$. Such a system is said to have *symmetry* if the Lagrangian $L : TQ \rightarrow \mathbb{R}$ is invariant under the natural lift to TQ of a proper left action of a Lie group G on Q . The Lagrangian determines a metric $\langle\langle \cdot, \cdot \rangle\rangle$ whose quadratic form is the kinetic energy, and thus G acts by isometries (that is, the metric is invariant under the action of G). The *momentum mapping* corresponding to the action of G on TQ is the map $\mathbf{J} : TQ \rightarrow \mathfrak{g}^*$ given by the formula

$$\mathbf{J}(v_q)(\xi) = \langle\langle v_q, \xi_Q(q) \rangle\rangle, \tag{2.1}$$

where ξ_Q is the infinitesimal generator of the action on Q corresponding to $\xi \in \mathfrak{g}$. Of course, Noether’s theorem guarantees that \mathbf{J} is conserved along solutions of the Euler–Lagrange equations. (See, for example, [7].)

The *locked inertia tensor* is defined to be the mapping $\mathbb{I} : Q \rightarrow \text{Lin}(\mathfrak{g}, \mathfrak{g}^*)$ given by

$$\langle\langle \mathbb{I}(q)\eta, \zeta \rangle\rangle := \langle\langle \eta_Q(q), \zeta_Q(q) \rangle\rangle = \mathbf{J}(\eta_Q(q))(\zeta),$$

where \langle, \rangle is the natural pairing of \mathfrak{g} and \mathfrak{g}^* . The name comes about from the fact that if one has, for example, two freely spinning rigid bodies connected by a ball-in-socket joint, then the locked inertia tensor at a configuration $q \in Q$ is the inertia tensor for the rigid body obtained by locking, or welding, the joint in this configuration. If $g \cdot q$ denotes the left action of $g \in G$ on q then the locked inertia tensor is equivariant in the sense that

$$\langle\langle \mathbb{I}(g \cdot q)\eta, \zeta \rangle\rangle = \langle\langle \mathbb{I}(q)\text{Ad}_{g^{-1}}\eta, \text{Ad}_{g^{-1}}\zeta \rangle\rangle.$$

Notice in particular that if G is abelian then \mathbb{I} is invariant under the group action.

The *angular velocity of the locked system* is defined via the map

$$\alpha : TQ \rightarrow \mathfrak{g} : v_q \mapsto \mathbb{I}^{-1}(q)\mathbf{J}(v_q).$$

The map α is a connection on the bundle $Q \rightarrow Q/G$ known as the *mechanical connection*. It is useful to recall the *horizontal–vertical decomposition* of a vector $v \in T_q Q$ given by the prescription

$$v = \text{hor}_q v + \text{vert}_q v,$$

where $\text{vert}_q v = [\alpha(v_q)]_Q(q)$ and $\text{hor}_q v = v - \text{vert}_q v$. One may think of $\text{vert}_q v$ as the orthogonal projection of $v \in T_q Q$.

Let $(q_e, \dot{q}_e) \in TQ$ and $\mu := \mathbf{J}(q_e, \dot{q}_e)$. By definition, (q_e, \dot{q}_e) is a *relative equilibrium* if there is a $\xi \in \mathfrak{g}$ such that the solution curve in TQ passing through (q_e, \dot{q}_e) is given by the one-parameter family

$$t \mapsto \exp(t\xi) \cdot (q_e, \dot{q}_e).$$

Observation 2.1. *If $z_e := (q_e, \dot{q}_e) \in TQ$ is a relative equilibrium, then so is the left translation $g \cdot z_e$ for any $g \in G$. More precisely, if $z(t) = \exp(t\xi) \cdot z_e$ is a relative equilibrium dynamical orbit, then so is $g \cdot z(t) = \exp[(\text{Ad}_g \xi)t] \cdot (g \cdot z_e)$.*

For a proof, see [7].

A useful quantity for identifying relative equilibria is the *augmented potential*, defined by

$$V_\xi(q) := V(q) - \frac{1}{2} \langle \mathbb{I}(q)\xi, \xi \rangle.$$

A point $(q_e, \xi_Q(q_e)) \in TQ$ is a relative equilibrium if and only if q_e is a critical point of V_ξ and $\dot{q}_e = \xi_Q(q_e)$. (See [7] for a proof.) Observe that if $(q_e, \xi_Q(q_e))$ is a relative equilibrium, then the unique solution curve through that point is given by $\exp(t\xi) \cdot (q_e, \dot{q}_e)$. Thus every point on that curve is also a relative equilibrium. Note that at a relative equilibrium, the horizontal part of the velocity vector is null. That is, for all t along the relative equilibrium $(q(t), \dot{q}(t)) := \exp(t\xi) \cdot (q_e, \dot{q}_e)$, we have

$$\dot{q}(t) = \text{vert}_{q(t)} \dot{q}(t) = \xi_Q(q(t)).$$

2.2. The Refined Saari Problem

Hernández-Garduño et al. [5] point out that a relative equilibrium curve $q(t)$ does not necessarily conserve the locked inertia tensor $\mathbb{I}(q(t))$. Instead, they prove that for any simple mechanical system, a relative equilibrium curve $q(t) = \exp t\xi \cdot q_e$ conserves the locked inertia tensor along ξ , that is, $\mathbb{I}(q(t))\xi$ is constant as a curve in \mathfrak{g}^* . This observation leads to a *Refined Saari Problem* in the context of simple mechanical systems:

Find classes of simple mechanical systems with symmetry such that a solution $q(t)$ of the Euler–Lagrange equations is a relative equilibrium if and only if $\mathbb{I}(q(t))\xi$ is constant as a curve in \mathfrak{g}^ , where $\xi_Q(q(0)) = \dot{q}(0)$.*

One such class is that of mechanical systems on Lie groups with no potential function, precisely, the class of mechanical systems with the configuration manifold being the group itself and with no potential function [5]. For $G = SO(3)$ this result becomes the well-known fact that a necessary and sufficient condition for relative equilibrium of a rigid body in \mathbb{R}^3 is the alignment of the angular velocity and the angular momentum.

3. Slice coordinates around a non-symmetric point

For the Refined Saari Problem in the case of a simple mechanical system on a general configuration manifold Q we invoke the technique of slices. In this section we shall cast a simple mechanical system in a slice parametrization, state a reduction of Hamilton’s principle, and express conditions for relative equilibria in the slice coordinates.

3.1. Slice coordinates

The idea behind slices is to locally decompose motion into two coupled systems. One describes the dynamics in the symmetry direction, or “rigid body” motion on TG . The other describes the dynamics in the direction orthogonal to the symmetry, or “shape-vibrational” motion on TS where S is chosen to be a submanifold of Q orthogonal to the group orbit. For instance, for a planar three-unit-mass particle system initially disposed in a triangular configuration, the motion in the group direction describes the changes in the angular velocity and the rotation of the triangle, whereas the dynamics on TS catches changes of the triangle’s shape. We briefly present here the formal background needed to develop the above ideas. For more detailed coverage of these topics, see [3,13,16], or [19].

Let Q be a smooth manifold and G a Lie group acting smoothly and properly on Q on the left. The *isotropy subgroup* of a point $q \in Q$ is $G_q := \{g \in G | gq = q\}$. By definition, the action is free if all of the isotropy subgroups

are trivial. A point $q_0 \in Q$ is called *non-symmetric* if it has trivial isotropy, that is, if $G_{q_0} = e$. Now consider a non-symmetric point $q_0 \in Q$ and S a manifold together with the left action on of G on $G \times S$ given by

$$G \times (G \times S) \longrightarrow (G \times S) : (h, (g, s)) \longmapsto (hg, s). \tag{3.1}$$

This action is free and proper. A *tube* for the G action at q_0 is a G -equivariant diffeomorphism from the product $G \times S$ to an open neighborhood of $G \cdot q_0$ in Q that, for some $s_0 \in S$, it maps (e, s_0) to q_0 . The space S may be embedded in $G \times S$ as $\{(e, s) : s \in S\}$; the image of S by the tube is called a *slice*.

The slice theorem of Palais [14] states that tubes always exist for smooth proper actions of a Lie group G on a manifold Q . If $(Q, \mathbb{K}(\cdot, \cdot))$ is a smooth Riemannian manifold and the action of G is linear, the theorem can be stated as follows: Given the non-symmetric point $q_0 \in Q$, let N be the orthogonal complement to $\mathfrak{g} \cdot q_0$, the tangent of the group orbit. Then there exists a neighborhood S of 0 in N such that the map

$$\tau : G \times S \rightarrow Q : (g, s) \rightarrow g \cdot (q_0 + s) \tag{3.2}$$

is a tube for the G action at q_0 . The complement $N = (\mathfrak{g} \cdot q_0)^\perp$ is called a *linear slice* to the G action at q_0 . The product $G \times N$ may be identified with the normal bundle of the orbit $G \cdot q_0$. Since the G action is linear, S may be chosen to be any neighborhood of 0 such that τ is injective. Note that TS is trivial, as S is a subset of a vector space. Also, $T_s S \cong N$ for any $s \in S$. We write elements of TS as (s, \dot{s}) . The tangent bundle $T(G \times S) \simeq TG \times TS$ is identified with $G \times \mathfrak{g} \times TS$ via the left trivialization

$$G \times \mathfrak{g} \times TS \longrightarrow TG \times TS : (g, \xi, s, \dot{s}) \longmapsto (g, T_e L_g \xi, s, \dot{s}). \tag{3.3}$$

For any $(g, s) \in G \times S$, the map $T_{(g,s)}\tau$ can be identified with an isomorphism from $\mathfrak{g} \times T_s S$ to $T_{\tau(g,s)}Q$. A velocity vector $v_q \in TQ$ is represented by the tangent map

$$T\tau : G \times \mathfrak{g} \times TS \longrightarrow TQ : (g, \xi, s, \dot{s}) \longrightarrow g \left(\xi_Q(q_0 + s) + \dot{s} \right). \tag{3.4}$$

3.2. The Lagrangian in slice coordinates

In the special case of a simple mechanical system, the Lagrangian $L : TQ \rightarrow \mathbb{R}$ has the form

$$L(q, v_q) = \frac{1}{2} \mathbb{K}(v_q, v_q) - V(q)$$

for some G -invariant Riemannian metric \mathbb{K} on Q , called the *kinetic energy*, and some G -invariant potential $V : Q \rightarrow \mathbb{R}$. We shall compute L for such systems, using the results of the previous section.

Let $N := (\mathfrak{g} \cdot q_0)^\perp$ be the orthogonal complement to the tangent to the group orbit through q_0 , with respect to the given metric. Using parametrization (3.4), recall that for any $(g, s) \in G \times S$, the map $T_{(g,s)}\tau$ can be identified with an isomorphism from $\mathfrak{g} \times N$ to $T_{\tau(g,s)}Q$. We write the metric tensor in these coordinates. Since the metric is G -invariant, $\mathbb{K}(\tau(g, s))$ depends only on s . For any s , we see that $\mathbb{K}(s)$ is a symmetric bilinear form on $\mathfrak{g} \times N$, which we represent as a matrix. This matrix can be written in block form, with respect to the splitting $T_{\tau(g,s)}Q \cong \mathfrak{g} \times N$, as follows:

$$\mathbb{K}(s) = \begin{pmatrix} \mathbb{I}_B(s) & \mathbb{C}(s) \\ \mathbb{C}^T(s) & m(s) \end{pmatrix}.$$

The block \mathbb{I}_B is called the *body locked inertia tensor*. It is related to the usual locked inertia tensor \mathbb{I} by

$$\langle \mathbb{I}(g \cdot (q_0 + s)) \xi, \eta \rangle = \langle \mathbb{I}_B(s) \text{Ad}_{g^{-1}} \xi, \text{Ad}_{g^{-1}} \eta \rangle \tag{3.5}$$

for any $\xi, \eta \in \mathfrak{g}$. The block $m(s)$ is called the *reduced mass*. The terminology comes from the fact that the kinetic energy matrix is often called the mass matrix. Note that $\mathbb{I}_B(s)$ and $m(s)$ are invertible. The block $\mathbb{C}(s)$ is called the *Coriolis tensor*. It couples the system and is related to the usual Coriolis forces. Our choice of coordinates $q_0 = \tau(e, 0)$ enforces $\mathbb{C}(0) = 0$, since $T_{(e,0)}\tau$ maps $\mathfrak{g} \times \{0\}$ to $\mathfrak{g} \cdot q_0$ and $\{0\} \times N$ to $T_{q_0}(q_0 + N) = N = (\mathfrak{g} \cdot q_0)^\perp$. Therefore the mechanical system in slice coordinates is decoupled at q_0 .

Since it is G -invariant, the potential V is written in slice coordinates as $V(s)$. So, L takes the form

$$L_{\text{slice}}(g, \xi, s, \dot{s}) = L \circ (T\tau)(g, \xi, s, \dot{s}) = \frac{1}{2} (\xi \quad \dot{s}) \mathbb{K}(s) \begin{pmatrix} \xi \\ \dot{s} \end{pmatrix} - V(s). \quad (3.6)$$

Using (2.1) and the identity $\xi_Q(g \cdot q) = (\text{Ad}_{g^{-1}}\xi)_Q(q)$, we may express the momentum map in slice coordinates:

$$\mathbf{J} : G \times \mathfrak{g} \times TS \longrightarrow \mathfrak{g}^* : (g, \xi, s, \dot{s}) \longmapsto \text{Ad}_g^*(\mathbb{I}_B(s)\xi + \mathbb{C}(s)\dot{s}).$$

3.3. The Euler–Poincaré–Lagrange system

We begin by observing that the tangent lift of (3.1) acts only on the “rigid body” component $TG \simeq G \times \mathfrak{g}$ of the phase space $TG \times TS \simeq G \times \mathfrak{g} \times TS$. Since the Lagrangian is left invariant under the action of G , we are able to reduce the dynamics using Hamilton’s variational principle adapted to our particular phase space structure. The reduced equations of motion form a coupled system formed by a reduced Euler–Poincaré rigid body part and a (tangent bundle) Lagrangian shape-vibrational part. (See [8] or [9] for a summary of Euler–Poincaré reduction of Lagrangian mechanics.)

Theorem 3.1. *Let G be a Lie group and S be a vector space. Consider a free and proper left action of G on $G \times S$. Also, let $L : TG \times TS \rightarrow \mathbb{R}$ be a left-invariant Lagrangian and its restriction to $e \in G$ be $l : \mathfrak{g} \times TS \rightarrow \mathbb{R}$. For a curve $(g(t), s(t)) \in G \times S$ let $(\xi(t), \dot{s}(t)) = (g(t)^{-1}\dot{g}(t), \dot{s}(t))$, i.e., $\xi(t) = T_{g(t)}L_{g(t)^{-1}}\dot{g}(t)$. Then the following are equivalent:*

(i) *Hamilton’s principle*

$$\delta \int_a^b L(g(t), \dot{g}(t), s(t), \dot{s}(t)) = 0$$

holds for variations $(\delta g(t), \delta s(t))$ vanishing at the end points.

(ii) *The curve $(g(t), s(t))$ satisfies the Euler–Lagrange equations of L .*

(iii) *The variational principle*

$$\delta \int_a^b l(\xi(t), \dot{s}(t)) = 0$$

holds on $\mathfrak{g} \times TS$ using variations of the form

$$(\delta \xi, \delta \dot{s}) = (\dot{\eta} + [\xi, \eta], \delta \dot{s}) \in T\mathfrak{g} \times TTS$$

where η vanishes at the end points.

(iv) *The Euler–Poincaré–Lagrange (EPL) coupled equations hold:*

$$\begin{cases} \frac{d}{dt} \frac{\delta l}{\delta \xi} = \text{ad}_\xi^* \frac{\delta l}{\delta \xi} \\ \frac{d}{dt} \frac{\delta l}{\delta \dot{s}} = \frac{\delta l}{\delta \dot{s}}. \end{cases}$$

Proof. This is directly analogous to the proof for the Euler–Poincaré equations (see, for instance, [9]). \square

Observation 3.2. *If TS is trivial then the EPL equations reduce to the well-known Euler–Poincaré equations on \mathfrak{g} .*

For a simple mechanical system with Lagrangian (3.6), the EPL coupled system takes the form:

$$\frac{d}{dt} (\mathbb{I}_B(s)\xi + \mathbb{C}(s)\dot{s}) = \text{ad}_\xi^* (\mathbb{I}_B(s)\xi + \mathbb{C}(s)\dot{s}) \quad (3.7)$$

$$\frac{d}{dt} (\mathbb{C}^T(s)\xi + m(s)\dot{s}) = \frac{\partial}{\partial s} \left(\frac{1}{2} \langle \mathbb{I}_B(s)\xi, \xi \rangle + \langle \mathbb{C}(s)\dot{s}, \xi \rangle + \frac{1}{2} \langle m(s)\dot{s}, \dot{s} \rangle - V(s) \right). \quad (3.8)$$

The dynamical picture of the system is completed by the reconstruction equation

$$\dot{g}(t) = g(t)\xi(t), \quad g(0) = e. \tag{3.9}$$

Given a solution $(\xi(t), s(t), \dot{s}(t))$ of the reduced EPL system, the orbit in the full unreduced space is retrieved by integrating the reconstruction equation. This is a standard procedure for systems with symmetry.

3.4. Relative equilibria in slice coordinates

Relative equilibria are special solutions that are characterized by constant motion along the orbits of the symmetry group. In a slice parametrization, locally factoring out the action of G on the phase space, the quotient space is $\mathfrak{g} \times TS$. It follows that relative equilibria correspond to equilibria in the reduced space together with constant velocity in the group (i.e., in (3.9) $\xi(t)$ is constant).

To simplify the notation, from now on we write the (local) diffeomorphism (3.2) $q = g(q_0 + s)$ as $q \simeq (g, s)$ and its tangent lift (3.4) as $(q, v) \simeq (g, \xi, s, \dot{s})$. Also, we define $\tilde{V} := V \circ \tau$ and $\tilde{\mathbb{I}} := \mathbb{I} \circ \tau$.

Recall that in the case of a simple mechanical system, a point $(q_e, v_e) \in TQ$ is a relative equilibrium if and only if there is a $\xi \in \mathfrak{g}$ such that

- (i) $v_e = \xi_Q(q_e)$, and
- (ii) q_e is a critical point of the augmented potential $V_\xi(q) := V(q) - \frac{1}{2} \langle \mathbb{I}(q)\xi, \xi \rangle$.

In slice coordinates these conditions $G \times \mathfrak{g} \times TS$ become:

Proposition 3.3. *A point $(g_e, \xi_e, s_e, \dot{s}_e) \in G \times \mathfrak{g} \times TS$ is a relative equilibrium if and only if*

- (i) $\dot{s}_e = 0$, and
- (ii) (g_e, s_e) is a critical point of the augmented potential in slice coordinates,

$$\tilde{V}_\xi(g, s) := \tilde{V}(s) - \frac{1}{2} \langle \tilde{\mathbb{I}}(g, s)\xi, \xi \rangle$$

where $\xi = \text{Ad}_{g_e} \xi_e$.

Proof. By identification of the infinitesimal generators $\xi_Q(q)$ and $\xi_{G \times S}(g, s)$ we have

$$\xi_{G \times S}(g, s) = (\xi_G(g), 0),$$

where $\xi_G(g)$ is the infinitesimal generator corresponding to the left action of G on itself. It follows that $\xi_G(g_e) = (g_e, \xi_e)$ and $\dot{s}_e = 0$. Via the left trivialization (3.3), we have $\xi_G(g_e) = (g_e, \text{Ad}_{g_e^{-1}}\xi)$ and therefore $\xi = \text{Ad}_{g_e}\xi_e$. Condition (ii) is obtained by a direct substitution of q with $\tau(g, s)$ in the augmented potential formula. \square

Observation 3.4.

$$\frac{\partial}{\partial g} \Big|_{g=g_e} \frac{1}{2} \langle \tilde{\mathbb{I}}(g, s)\xi, \xi \rangle = 0 \quad \text{if and only if} \quad \text{ad}_\xi^* (\tilde{\mathbb{I}}(g_e, s)\xi) = 0.$$

Proof. Any vector $\delta g \in T_{g_e}G$ may be expressed as $\delta g = \eta_G(g_e)$ for some $\eta \in \mathfrak{g}$. If $\frac{\partial}{\partial g} \Big|_{g=g_e} \frac{1}{2} \langle \tilde{\mathbb{I}}(g, s)\xi, \xi \rangle = 0$ we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial g} \Big|_{g=g_e} \frac{1}{2} \langle \tilde{\mathbb{I}}(g, s)\xi, \xi \rangle \delta g = \frac{\partial}{\partial g} \Big|_{g=g_e} \frac{1}{2} \langle \tilde{\mathbb{I}}(g, s)\xi, \xi \rangle \eta_G(g_e) \\ &= \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \langle \tilde{\mathbb{I}}(\exp(t\eta)g_e, s)\xi, \xi \rangle \\ &= \frac{d}{dt} \Big|_{t=0} \frac{1}{2} \langle \tilde{\mathbb{I}}(g_e, s) \text{Ad}_{\exp(-t\eta)}\xi, \text{Ad}_{\exp(-t\eta)}\xi \rangle \\ &= \langle \tilde{\mathbb{I}}(g_e, s)\xi, \text{ad}_\xi \eta \rangle = \langle \text{ad}_\xi^* (\tilde{\mathbb{I}}(g_e, s)\xi), \eta \rangle. \end{aligned}$$

Thus, $\text{ad}_\xi^* (\tilde{\mathbb{I}}(g_e, s)\xi) = 0$. The argument is completely reversible. \square

Lemma 3.5. $(g_e, \xi_e, s_e, \dot{s}_e) \in G \times \mathfrak{g} \times TS$ is a relative equilibrium if and only if all of the following are satisfied:

- (i) $\dot{s}_e = 0$,
- (ii) s_e is a critical point of the augmented potential given by

$$\tilde{V}_{\xi}^{g_e}(s) := \tilde{V}(s) - \frac{1}{2} \langle \tilde{\mathbb{I}}(g_e, s)\xi, \xi \rangle$$

and

- (iii) $\text{ad}_{\xi}^* (\tilde{\mathbb{I}}(g_e, s_e)\xi) = 0$, where $\xi = \text{Ad}_{g_e} \xi_e$.

Proof. Using Proposition 3.3 and Observation 3.4, the verification is immediate. \square

By Observation 2.1 the left translation of a relative equilibrium is a relative equilibrium. Therefore, without losing generality, we may choose g_e to be the symmetry group identity. Consequently, $\xi = \xi_e$ and we have the following criterion.

Criterion 3.6. In a slice parametrization, a relative equilibrium is a dynamical orbit of the form $(g(t), \xi_e, s_e, \dot{s}_e)$, where (ξ_e, s_e, \dot{s}_e) satisfies

- (i) $\dot{s}_e = 0$,
- (ii) s_e is a critical point of the augmented potential given by

$$\tilde{V}_{\xi_e}(s) := \tilde{V}(s) - \frac{1}{2} \langle \mathbb{I}_B(s)\xi_e, \xi_e \rangle, \quad \text{and} \tag{3.10}$$

- (iii) $\text{ad}_{\xi_e}^* (\mathbb{I}_B(s_e)\xi_e) = 0$,

and $g(t)$ is the solution of the reconstruction equation $\dot{g}(t) = g(t)\xi_e$, $g(0) = e$.

4. Constant locked inertia tensor and Saari’s Conjecture

4.1. Constant locked inertia tensor and relative equilibria

The aim of this subsection is to establish conditions on the locked inertia tensor that suffice to make a solution a relative equilibrium. We also discuss the Refined Saari Problem in slice coordinates.

Proposition 4.1. A dynamical orbit $(g(t), \xi(t), s(t), \dot{s}(t))$ with an initial condition $(e, s_0, \xi_0, \dot{s}_0)$ is a relative equilibrium if and only if all of the following are satisfied.

- (i) $\dot{s}_0 = 0$,
- (ii) s_0 is a critical point of $\tilde{V}_{\xi_0}(s)$ given in (3.10), and
- (iii) $\left. \frac{d}{dt} \right|_{t=0} \tilde{\mathbb{I}}(g(t), s(t))\xi_0 = 0$, that is,

$$\left. \frac{d}{dt} \right|_{t=0} \langle \tilde{\mathbb{I}}(g(t), s(t))\xi_0, \eta \rangle = 0 \quad \text{for all } \eta \in \mathfrak{g}.$$

Proof. Items (i) and (ii) appear verbatim from Criterion 3.6. If the curve $(g(t), \xi(t), s(t), \dot{s}(t))$ is a relative equilibrium, then (iii) is the restatement in slice coordinates of Proposition 5.1 in [5]. Conversely, it suffices to verify condition (iii) of Criterion 3.6. Since $\tilde{\mathbb{I}}(g(t), s(t))\xi_0$ is constant in time, we apply (3.5), yielding

$$\left. \frac{d}{dt} \right|_{t=0} \langle \text{Ad}_{g^{-1}(t)}^* \mathbb{I}_B(s(t)) \text{Ad}_{g^{-1}(t)} \xi_0, \eta \rangle = 0 \quad \text{for all } \eta \in \mathfrak{g},$$

or

$$\left. \frac{d}{dt} \right|_{t=0} \langle \mathbb{I}_B(s(t)) \text{Ad}_{g^{-1}(t)} \xi_0, \text{Ad}_{g^{-1}(t)} \eta \rangle = 0 \quad \text{for all } \eta \in \mathfrak{g}.$$

Continuing, we get

$$\begin{aligned} & \left\langle \frac{d}{dt} \Big|_{t=0} \mathbb{I}_B(s(t)) \text{Ad}_{g^{-1}(0)} \xi_0, \text{Ad}_{g^{-1}(0)} \eta \right\rangle + \langle \mathbb{I}_B(s(0)) \text{Ad}_{g^{-1}(0)} \text{ad}_{(-\xi_0)} \xi_0, \text{Ad}_{g^{-1}(0)} \eta \rangle \\ & + \langle \mathbb{I}_B(s(0)) \text{Ad}_{g^{-1}(0)} \xi_0, \text{Ad}_{g^{-1}(0)} \text{ad}_{\xi_0} \eta \rangle = 0 \end{aligned} \tag{4.1}$$

where we used the fact that

$$\frac{d}{dt} \Big|_{t=0} \text{Ad}_{g^{-1}(t)} \zeta = \frac{d}{dt} \Big|_{t=0} \text{Ad}_{g^{-1}(0) \exp(-t\xi_0)} \zeta \quad \text{for all } \zeta \in \mathfrak{g}.$$

Using (i), relation (4.1) becomes

$$\langle \mathbb{I}_B(s_0) \xi_0, \text{ad}_{\xi_0} \eta \rangle = 0 \quad \text{for all } \eta \in \mathfrak{g}.$$

Finally, $g(t)$ may be recovered via the reconstruction equations (3.9). \square

Observation 4.2. Condition (iii) of Proposition 4.1 can be replaced with the more restrictive condition that $\tilde{\mathbb{I}}(g(t), s(t)) \xi_0$ is constant along the dynamical orbit. Necessity of the replacement condition is a direct consequence of Noether’s theorem, as shown in Proposition 5.1 in [5].

Observation 4.3. Condition (i) implies that $\frac{d}{dt} \Big|_{t=0} \langle \mathbb{I}_B(s(t)) \xi, \eta \rangle = 0$ for all $\xi, \eta \in \mathfrak{g}$, or, more restrictively, $\mathbb{I}_B(s(t))$ is constant along the orbit.

Observation 4.4. One may easily verify that Proposition 5.2 in [5] is generalized by Criterion 3.6. Indeed, Proposition 5.2 in [5] refers to the specific case of Q being the symmetry group G itself. This corresponds to a trivial slice, i.e., $Q = G \times \{0\}$, and thus conditions (i) and (ii) of Criterion 3.6 are satisfied vacuously. The condition from Proposition 5.2 in [5] that for any $\eta \in \mathfrak{g}$ the quantity $\langle \mathbb{I}(g(t), 0) \xi, \eta \rangle$ be constant then satisfies condition (iii) of Criterion 3.6.

Observation 4.5. In the planar N -body problem, $Q = \mathbb{R}^{3N} \setminus \{\text{collisions}\}$ and $G = SO(2)$. In this case the action is free, since the only non-symmetric point $q = 0$ corresponds to a total collision. Choose $q_0 \in Q$ as a base point for a slice parametrization. Near $q_0 \simeq (e, 0)$ we have $TQ \simeq SO(2) \times so(2) \times TS$ where S is an open subset of \mathbb{R}^{2N-2} . Since $SO(2)$ is abelian, $\tilde{\mathbb{I}}(g, s) = \mathbb{I}_B(s)$. Now consider a dynamical orbit $(q(t), v(t)) \simeq (g(t), \xi(t), s(t), \dot{s}(t))$ with initial condition $(q_0, v_0) \simeq (e, \xi_0, s_0, \dot{s}_0)$ such that the inertia tensor $\mathbb{I}(q(t))$ is constant. Because $\mathbb{I}(q(t)) = \mathbb{I}_B(s(t))$, by applying Proposition 4.1 this orbit is a relative equilibrium if and only if $\dot{s}_0 = 0$ and s_0 is a critical point of the augmented potential $V_{\xi_0}(s)$.

We return now to the Refined Saari Problem restated in slice coordinates:

Consider a simple mechanical system with symmetry. Let $q(t) \simeq (g(t), s(t))$ be a solution of the EPL equations together with initial conditions $(q(0), \dot{q}(0)) \simeq (e, \xi_0, 0, \dot{s}_0)$, and the reconstruction equation. Find classes of simple mechanical systems with symmetry such that $q(t) \simeq (g(t), s(t))$ is a relative equilibrium if and only if $\tilde{\mathbb{I}}(g(t), s(t)) \xi_0$ is constant as a curve in \mathfrak{g}^* .

Using Proposition 4.1 and Observations 4.2 and 4.3, we proceed to identify one such class of solutions of the Refined Saari Problem.

Theorem 4.6. A class of solutions of the Refined Saari Problem consists of systems with the following properties:

- (i) the configuration space is a finite dimensional Riemannian manifold Q with symmetry from a free and proper action of a k -dimensional Lie group G ;
- (ii) the matrix $\left[\frac{\partial \mathbb{I}_B}{\partial s} \right]$ has maximal rank for all $s \in S$ and $\frac{k(k+1)}{2} \geq n - k$, where n is the dimension of Q (note that $n - k$ is the dimension of the slice S), and
- (iii) $\mathbb{I}_B(s(t))$ is constant, that is, for each $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{g}$,

$$\langle \mathbb{I}_B(s(t)) \xi, \eta \rangle = C_{(\xi, \eta)},$$

where $C_{(\xi, \eta)}$ is a constant depending on ξ and η .

Proof. Since $\mathbb{I}_B(s(t))$ is constant, we have $\left[\frac{\partial \mathbb{I}_B}{\partial s}\right] \dot{s} = 0$. Because the matrix $\left[\frac{\partial \mathbb{I}_B}{\partial s}\right]$ has maximal rank for all $s \in S$ and it is symmetric, its rank equals $\max\{k(k+1)/2, n-k\}$. Given that $k(k+1)/2 \geq n-k$, it follows that the linear system $\left[\frac{\partial \mathbb{I}_B}{\partial s}\right] \dot{s} = 0$ admits only the trivial solution $\dot{s} = 0$, and this happens for all $t \geq 0$. Thus $\dot{s}(t) = 0$, and so $s(t) = s(0) = 0$ for all $t \geq 0$.

Our statement assumes that $(g(t), \xi(t), s(t), \dot{s}(t))$ is a dynamical orbit. In particular, this means that $(\xi(t), s(t), \dot{s}(t))$ solves the EPL system. Substituting $s(t) = 0 = \dot{s}(t)$ into Eq. (3.7), we obtain that

$$\frac{d}{dt}(\mathbb{I}_B(0)\xi(t)) = \text{ad}_{\xi(t)}^*(\mathbb{I}_B(0)\xi(t)), \tag{4.2}$$

with initial condition $\xi(0) = \xi_0$.

Note that condition (ii) in the hypothesis implies that for each $\eta \in \mathfrak{g}$, $\left.\frac{d}{dt}\right|_{t=0} \langle \mathbb{I}_B(s(t))\xi_0, \eta \rangle = 0$. Following the same reasoning as in Proposition 4.1, if $\tilde{\mathbb{I}}(g(t), s(t))$ is constant then $\text{ad}_{\xi_0}^*(\mathbb{I}_B(s_0)\xi_0) = 0$. From uniqueness of solutions, the solution of Eq. (4.2) is $\xi(t) = \xi_0$.

Recall that at the base of the slice we have the coupling term $C(0) = 0$. Since $s(t) = 0$, it follows that $C(s(t)) = 0$. Rewriting (3.8) using all information gathered, we obtain that

$$0 = \frac{\partial}{\partial s} \left(\frac{1}{2} \langle \mathbb{I}_B(0)\xi_0, \xi_0 \rangle - V(s) \right),$$

i.e., $s = 0$ is a critical point of $V_{\xi_0}(s)$. Thus, all three conditions of Criterion 3.6 are satisfied, so the solution is a relative equilibrium.

Conversely, if the solution is a relative equilibrium then by Observation 4.2 it follows that $\tilde{\mathbb{I}}(g(t), s(t))\xi_0$ is constant. \square

It is natural to ask how restrictive the maximal rank condition is. Since $\mathbb{I}_B(s) = \mathbb{I}(q)$ for $q \simeq (e, s)$, we may ask a more general question: What are the critical points of $\mathbb{I}(q)$? For linear actions on inner product spaces we have a response.

Proposition 4.7. Consider a linear action by isometries of a Lie group G on an inner product space W . Write $\xi \cdot q := \xi_W(q)$ and denote by $\xi \cdot v$ the infinitesimal action of $\xi \in \mathfrak{g}$ on $v \in T_q W \cong W$. Let q_0 be a critical point for the locked inertia tensor. Then $\xi \cdot (\xi \cdot q_0) = 0$.

Proof. Let q_0 be a critical point for the locked inertia tensor. Then

$$0 = D_{q_0} \langle \mathbb{I}(q)\xi, \eta \rangle = D_{q_0} \langle \xi \cdot q, \eta \cdot q \rangle_W \quad \forall \xi, \eta \in \mathfrak{g}.$$

For a smooth curve $q(t)$ in W such that $q(0) = q_0$ and $\dot{q}(0) = \delta q$,

$$\left.\frac{d}{dt}\right|_{t=0} \langle \xi \cdot q(t), \eta \cdot q(t) \rangle_W = D_{q_0} \langle \xi \cdot q, \eta \cdot q \rangle_W \cdot \delta q = 0,$$

so

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \langle \xi \cdot q(t), \eta \cdot q(t) \rangle_W \\ &= \langle \xi \cdot \delta q, \eta \cdot q_0 \rangle_W + \langle \xi \cdot q_0, \eta \cdot \delta q \rangle_W \\ &= \langle \delta q, -\xi \cdot (\eta \cdot q_0) \rangle_W + \langle -\eta \cdot (\xi \cdot q_0), \delta q \rangle_W \\ &= -\langle \delta q, \xi \cdot (\eta \cdot q_0) + \eta \cdot (\xi \cdot q_0) \rangle_W, \end{aligned}$$

using the definition of the infinitesimal action of \mathfrak{g} on Q and the G -invariance of the metric. Since the last relation is valid for all $\delta q \in T_{q_0} W$ we get $\xi \cdot (\eta \cdot q_0) + \eta \cdot (\xi \cdot q_0) = 0 \forall \xi, \eta \in \mathfrak{g}$. Let $\xi = \eta$ and the result follows immediately. \square

Corollary 4.8. Consider the $SO(3)$ diagonal action on $W = \mathbb{R}^{3n}$. Let $m_i, i = 1, 2, \dots, n$, be some strictly positive numbers and endow W with the constant metric $\langle v, w \rangle_W := (1/2)v^T M w$, where M is the constant mass matrix $M = (M_{k,l})_{k,l=1,2,\dots,3n}$, $M_{k,l} = 0$ for $k \neq l$ and $M_{k,k} = m_i$ for $k = 3i - 2, 3i - 1, 3i$ with $i = 1, 2, \dots, n$. Then $q = 0$ is the only critical point of the locked inertia tensor.

Proof. Let $q = (q_1, q_2, \dots, q_n) \in W$, where each $q_i \in \mathbb{R}^3$, $i = 1, 2, \dots, n$. The diagonal action of $SO(3)$ on W is given by

$$(g, q) \mapsto g \cdot q := (g \cdot q_1, g \cdot q_2, \dots, g \cdot q_n),$$

that is, each component q_i of the vector q is rotated by $g \in SO(3)$. Consider q a critical point of \mathbb{I} . By Proposition 4.7, $\xi \cdot (\xi \cdot q) = 0$ for all $\xi \in so(3)$. Since the action is diagonal, we have that $\xi \cdot (\xi \cdot q_i) = 0$ for each $i = 1, 2, \dots, n$ and for all $\xi \in so(3)$.

Identifying $so(3)$ with \mathbb{R}^3 via the “hat” map (see, for instance, [8]), the last relation implies $\hat{\xi} \times (\hat{\xi} \times q_i) = 0$, for all $\hat{\xi} \in \mathbb{R}^3$. So for each $i = 1, 2, \dots, n$, $\hat{\xi}$ is parallel to $(\hat{\xi} \times q_i) = 0$ for all $\hat{\xi} \in \mathbb{R}^3$. But this is possible if and only if $q_i = 0$, $i = 1, 2, \dots, n$, i.e., if and only if $q = 0$. \square

4.2. Applications

Abelian symmetries

The Refined Saari Problem includes the subclass of simple mechanical systems with the following qualities:

- (1) the configuration manifold Q is an n -dimensional punctured inner product space $W \setminus \{0\}$;
- (2) the symmetry is given by a free linear action by isometries of an abelian Lie group G of dimension k ;
- (3) $\frac{k(k+1)}{2} \geq n - k$.

A generalized Saari’s Conjecture is true for the spatial three-body and four-body problems

Let G be the group of rotations in the space $SO(3)$ and consider the three-body problem. Using the Jacobi coordinates, the configuration space is $\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{(\mathbf{0}, \mathbf{0})\}$. (Recall that Jacobi coordinates are given by the relative vector between two of the masses and the position vector of the third mass relative to the center of mass of the first two.) The following generalization of Saari’s Conjecture is true for $N = 3$:

A solution $(q(t), \dot{q}(t))$ for the spatial N -body problem with $\mathbb{I}(q(t))\xi_0$ constant, where $(\xi_0)_Q q(0) = \dot{q}(0)$, and $\mathbb{I}_B(s(t)) = \text{constant}$, is a relative equilibrium.

Indeed, by Corollary 4.8, the locked inertia tensor \mathbb{I} has no critical points. If the masses are in a non-collinear configuration, then, by Theorem 4.6 (here $\dim G = \dim SO(3) = 3$, and $\dim Q = 6$), the above statement is true. If the masses are in a collinear configuration, then one may apply the results of Diacu et al. [4]. In conclusion, the generalized Saari’s Conjecture is true for the three-body problem in space, where the bodies may interact via any potential defined on the configuration space $\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{(\mathbf{0}, \mathbf{0})\}$.

A similar result applies to the spatial four-body problem. This is again a direct application of Theorem 4.6 with $\dim G = 3$ and $\dim Q = 9$ (where we have used the generalized Jacobi coordinates).

Constant locked inertia tensor trajectories for $SO(n)$ action on \mathbb{R}^n

Consider a simple mechanical system with configuration space $\mathbb{R}^n \setminus \{0\}$, $n \geq 2$, endowed with the usual inner product. Assume that the system has a symmetry given by the diagonal action of $SO(n)$ on \mathbb{R}^n . One may think about this problem as a generalized two-body problem in \mathbb{R}^n . Then a solution $(q(t), \dot{q}(t))$ with constant $\mathbb{I}(q(t))\xi_0$, where $(\xi_0)_Q q(0) = \dot{q}(0)$ and $\mathbb{I}_B(s(t))$ is constant, is a relative equilibrium.

Comments

If $\frac{k(k+1)}{2} < n - k$ then the constraints on the locked inertia tensor in Theorem 4.6 do not enforce $\dot{s}(t) = 0$. In this case, one may produce physically relevant counterexamples by taking advantage of the freedom of moving in the slice and choosing non-generic potentials. For instance, in [18], in a setting with $G = SO(2)$ and $S = \mathbb{R}^2$, the potential is chosen to be $V(s) := \mathbb{I}_B(s)\xi_0$. One can verify that the associated EPL system “decouples” i.e. $C(t) = 0$ for all t . Further, with the given choice for V , the system in $s = (s_1, s_2) \in S$ becomes two decoupled harmonic oscillators. It can be shown that the set of solutions preserving the moment of inertia form a large invariant manifold.

5. Conclusions

In the context of simple mechanical systems with symmetry, we have extended the augmented potential method of categorizing relative equilibria to a Palais slice at a non-symmetric base point. The new necessary and sufficient criteria for relative equilibrium are applied to the Refined Saari Problem to identify new classes of solutions.

For future investigation we may wish to consider relative equilibria at symmetric base points. Of course, if one point on a dynamical orbit is a relative equilibrium then every point is as well. So if such a curve passes through even one non-symmetric point then applying the criteria for relative equilibrium at the non-symmetric point will suffice to determine relative equilibrium for the entire curve. However, our method does not address solution curves for which every point on the curve is symmetric.

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